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Design of a questionnaire item to assess the acquisition of Van Hiele level 5 with regard to the definition process

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The renowned Van Hiele model of geometric reasoning proposes five progressive levels of development, ranging from level 1 (visual) to level 5 (rigor). The literature on the fifth level is relatively scarce, particularly in relation to questionnaires designed to measure the degree of acquisition of the four processes present in this level (definition, proof, recognition, and classification). In this paper, we present an item specifically focused on the definition process, with the aim of determining its validity in assessing the degree of acquisition of level 5. Following administration of the questionnaire in its present form, we have identified that this item is effective in analyzing the acquisition of level 5 with regards to certain aspects pertaining to the correlation between the definition of an object and the geometric context in which it is defined.

Keywords: Van Hiele model, geometry, definitions, questionnaires.

Introduction and objectives

In recent times, research in Mathematics Education has exhibited a growing interest in university-level studies. (Dibbs & Beach, 2017; Häusler, & Kuzle, 2022). The Van Hiele model has proven to be valuable tool for teaching and learning Geometry (Van Hiele 1986). In this context, it is necessary to expand on certain aspects of the model related to higher levels of reasoning. Arnal-Bailera and Manero (2023) have provided an in-depth description of the fifth level of the model showing the crucial role of definition and proof processes, these two processes had already been pointed out as two of the most important ones in advanced mathematical thought by Tall (1992). The next step is to give practical application to this knowledge by evaluating the extent to which Mathematics undergraduates have acquired this level of reasoning. A more comprehensive understanding of the acquisition of level 5 among Mathematics university students would facilitate the development of guidelines to enhance the teaching of processes that are found to be less acquired by them.

The current work is focused on an item of the questionnaire that we are currently designing to assess the acquisition of van Hiele's fifth level in Mathematics students at the grade level. The questionnaire is principally geared towards assessing the processes of proof and definition, the main processes in the fifth level. The item presented in this paper is specifically tailored to assess the definition process.

In this work we are trying to answer the following research question: Does the proposed questionnaire item adequately assess the degree of acquisition of level 5 regarding the definition process? The goals of the present study are twofold: firstly, to determine the extent to which the indicators of definition presented in Arnal-Bailera and Manero (2023) can be evaluated using the current questionnaire item, and secondly, to study the responses provided by students in terms of their attainment of van Hiele's fifth level relative to those indicators.

Theoretical framework

This section provides a summary of the Van Hiele model of geometric reasoning, with a focus on what is known about Level 5 and the definition process at this level.

In the 1950s, Pierre and Dina Van Hiele developed the Van Hiele model, which is considered one of the most significant theoretical frameworks in the teaching and learning of Geometry (Van Hiele, 1957). This model posits that there are five different levels of geometric reasoning, where different geometric concepts are used and understood differently (Hoffer, 1983; Burger & Shaughnessy, 1986; Van Hiele 1986; Jaime & Gutiérrez, 1990). The levels are sequential and hierarchical, implying that they are acquired in a specific order throughout the learning process.

The literature often describes the Van Hiele levels according to the different processes involved (De Villiers, 1987; Gutiérrez & Jaime, 1998; Burger & Shaughnessy, 1996). These processes include *definition* (use and formulation of definitions of geometrical objects), *proof* (convincing oneself or others of the truth of a statement), *classification* (sorting geometrical objects into different families or creating new groups to sort the objects), and *identification* (establishing the family to which a particular geometrical object belongs). A detailed description of levels 1 to 4 organized according to these processes can be found in Gutiérrez and Jaime (1998).

Arnal-Bailera and Manero (2023) noted that most of the related literature has focused on the development and study of the first four levels. This lack of research on the fifth Van Hiele level can be attributed to the fact that it is not related to the contents or abilities taught in school Geometry, as reasoning on different axiomatic systems is typically taught only in university Geometry courses. Historically, the emphasis of the academic community regarding mathematics education at the university level has been primarily on teacher training. However, in recent years, there has been a shift in focus towards the teaching and learning of Mathematics across a diverse range of university degree programs.

Of the few studies that have considered level 5, the works of Usiskin (1982), Mayberry (1983), and Blair (2004) are noteworthy. In Usiskin's work (1982), a test was designed to assess the Van Hiele level of students. This test comprised 25 questions, including some regarding level 5. Mayberry's study (1983) focused on the hierarchical structure of the levels and presented some guidelines for designing questions that correspond to each level. Specifically, Mayberry emphasized that questions at level 5 should involve propositions related to finite geometries, which is consistent with the questions proposed by Usiskin. More recently, Blair's doctoral dissertation (2004) also addressed level 5, describing tasks that involve non-conventional metrics, such as the Taxicab, as a potential means of developing level 5 acquisition.

Several researchers have explored high-level reasoning concerning the definition process. For instance, Martín-Molina et al. (2018) studied the practices of professional mathematicians when defining, placing this process within a broader context of generalization. Other authors, such as Larsen and Zandieh (2008), have described the motivations to create new definitions, pointing out that the proof process can be a motivation for defining when working across different geometries. Arnal-Bailera and Manero (2023) used the Delphi methodology to develop and validate indicators for the different processes involved in the fifth Van Hiele level. Specifically, they considered a panel of experts consisting of 25 University teachers (as proposed in the Delphi methodology), all of whom held a PhD in Geometry and Topology and different degrees of research experience, with a small group of the respondents (5 out of the 25) also having experience in Mathematics Education. Through the administration of a series of questionnaires and the analysis of the experts' answers, they obtained a list of indicators that were evaluated and elaborated in successive rounds to get a final list of five validated indicators (see Table 1).

Def1.	Constructs and uses definitions in different axiomatic systems.
Def2.	Understands that defining a given mathematical object is not absolute, but is an action relative to the geometric context in which one works, implying for example that the defined object may have different properties in each context.
Def3.	Defines new objects, for example, because it may be necessary to generalize existing ones or to prove a statement.
Def4.	Understands that a definition arises out of the necessity to introduce a new mathematical object or to emphasize a property.
Def5.	Compares equivalent definitions to choose the most interesting one, depending on the work to be done.

Table 1: Indicators validated describing the definition process (Arnal-Bailera & Manero, 2023)

Methods

This study employed a research and development approach to develop and validate an item for a questionnaire on the development of geometric thinking. This approach involved a sequential research process, in which design principles were established based on existing literature, a set of items was constructed, and presented to experts in the field to examine the validity of the structure. The feedback obtained from the experts was then utilized to develop a pilot version of the questionnaire, which was subsequently administered with a group of pre-service mathematics teachers. The final version of the questionnaire will consist of a list of items assessing all the five indicators of the definition process. Possibly one item could assess more than one indicator. Due to space reasons, we present only one of the items (see Figure 1). The item presented in this work has been chosen because it deals with a relevant aspect in many geometry courses at university level, namely the different definitions of objects depending on the geometrical context.

The items in our questionnaire were designed based on the structure proposed in previous studies (Gutiérrez & Jaime, 1998), where each item measures the acquisition of multiple levels, depending on the respondents' answers. This is achieved by presenting the same question with different hints in various tasks, which together form the item. For instance, as shown in Figure 1, item 4 consists of tasks 4.1, 4.2, and 4.3 (designed by the authors), with each task providing information on the respondent's acquisition of level 5, 4, or 3 respectively. We followed the criteria proposed by Creswell (2012) for constructing high-quality questions, including both closed- and open-ended questions and ensuring that questions are clearly understood by all participants. Furthermore, we conducted a pilot test of the questions, using the feedback obtained to refine and improve the questionnaire. With respect to the content of the questionnaire items, very few studies have focused on how to assess level

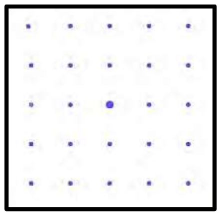
5. Among these studies, Blair's (2004) thesis is particularly noteworthy for its suggestion to use non-conventional metrics with classical questions. This approach was adopted in the present work.

In the 2020-21 academic year, the questionnaire was administered to a sample of 21 preservice teachers who held a degree in Mathematics (9) or Physics (12) and were enrolled in a Master's program in Spain aimed at preparing them to become secondary school teachers. The program included coursework related to the teaching and learning of geometry. Therefore, participants were expected to draw upon their reasoning skills to solve the tasks presented in the questionnaire.

ITEM 4. Normally to measure distances in the plane we use the Euclidean metric, which is defined as follows: Given two points, the distance between them is the length of the segment joining them. However, we can define other distances, such as the so-called postman's (or Taxicab) distance, which is defined as follows: the distance between two points is given by the shortest route joining those using only horizontal and vertical lines.

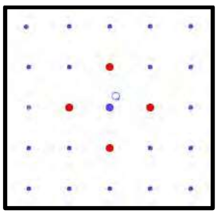
4.1 If we define the circumference as the set of points that are equidistant (at the same distance) from another point, what is the shape of the circumferences with the Taxicab metric? Justify your answer.

4.2 If we define a circumference as a set of points that are equidistant from another point, what shape do they have with the distance from the postman? Justify your answer. You can use the grid to draw the points that are equidistant from the point indicated.

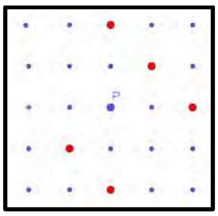


4.3 If we define a circle as a set of points that are equidistant (at the same distance) from another point. Look at the different drawings below:

All the marked points are, with the Taxicab metric, at distance __ from point Q.



All the marked points are, with the Taxicab metric, at distance __ from point P.



What is the shape of the circles with the Taxicab metric? Justify your answer.

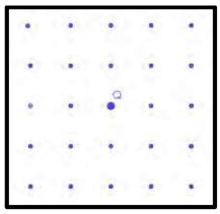


Figure 1: Item 4 statements

This item required students to determine the shape of a circumference using the Taxicab metric, which prompted them to reconsider their understanding of the definition and properties of a circumference with respect to the metric employed. Task 4.1 was designed without any hints or grid to support students' problem-solving process. In task 4.2, we provided a grid to facilitate construction of the circumference, while task 4.3 included different grids and closed-ended questions in the statement.

Participants were given 30 minutes to complete individually the three tasks which were to be answered in this particular order due to the fact that the statements or the drawings of tasks 4.2 and 4.3 contained relevant information to solve the previous tasks. Thus, each task was administered one at a time, with participants receiving the next task only upon completion and submission of the previous one. When a student considered that he/she had answered well in a previous section, he/she was allowed not to answer the following sections.

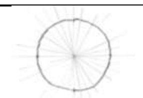
In this study, we present only an initial analysis of task 4.1. This should be followed at some point in the future by separate analyses of tasks 4.2 and 4.3 using a different set of indicators, as these tasks are aimed at assessing the acquisition of levels 4 and 3, respectively. Considering the design of the item and the indicators presented in Table 1, we conclude that this item contributes to the evaluation of def2 (understanding that defining a mathematical object is a relative action dependent on the geometric context). This is because, in task 4, the concept of the circumference is applied in a different geometric context due to the use of the Taxicab metric, which may result in different properties being associated with the defined object.

The degrees of acquisition of the fifth van Hiele level are based on Gutiérrez et al. (1991) who established 5 degrees: No acquisition, low, intermediate, high and complete acquisition. In our preliminary results we present examples of all of them.

Preliminary results

Both graduates in Mathematics and Physics showed a variety of levels of acquisition, although the most frequent among mathematicians was intermediate (4), while among physicists it was low (5).

In this section, we present and analyse five transcriptions of student responses to Task 4.1. to show examples of the five levels of acquisition. The first example shows no indicators of the definition process at the fifth level. The fifth example, in contrast, exhibits a thorough comprehension of the Taxicab metric. The rest of the examples represent the development of the fifth level. These examples have been categorized using the classification proposed by Gutiérrez et al. (1991), which designates the degrees of acquisition as "no acquisition" (4 students: 2 Mathematics. and 2 Physics), "low" (7: 2 and 5), "intermediate" (6: 4 and 2), "high" (3: 1 and 2) and "complete" (1 student: Physics).

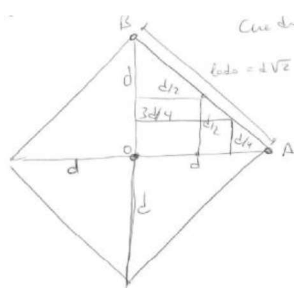


It is a polygon with the largest possible number of sides, so the angles will be as wide as possible and thus it will resemble a circumference.

Figure 2: student #2 answer, level 5 – no acquisition

Student #2 shows *no acquisition* of level 5 since his/her comments do not include any reference to the Taxicab metric. The answer of Student #7 (Figure 3, left) illustrates the influence of Euclidean Geometry on his/her reasoning process. The student mistakenly refers to the Taxicab metric by considering that segments can only be horizontal or vertical and, furthermore, tries to approximate the Euclidean circumference with them. Thus, the level of acquisition is classified as *low acquisition*.

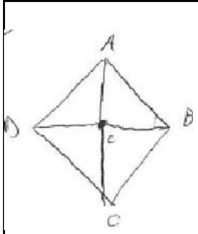
Circumference := set of points equidistant to another point.
With a high number of points in such a way that the horizontal or vertical line (Taxicab metric) that joins them was infinitesimal, the circumference would be appreciated analogously to the Euclidean one (…) the circumference would be similar to the Euclidean one.



(Circumferences) are squares, if d is the distance that is a constant. The side of the square is $\sqrt{2d^2} = d\sqrt{2}$

Figure 3: student #7, level 5 – low acquisition (left); student #15, level 5 – intermediate acquisition (right)

Student #15 exhibits a limited understanding of the Taxicab metric in his/her response (Figure 3, right). He/she fails to mention the specific orientation of the square that represents the circumference. Furthermore, instead of employing the Taxicab metric to determine the length of the square's side (which is 2d), the student calculates it using the Euclidean metric and the Pythagorean Theorem (resulting in $d\sqrt{2}$). Generally speaking, the student focuses on explaining the construction rather than providing explicit arguments to support it. However, by examining the center of the square, we can infer some implicit argumentation based on examples. Specifically, the student employs two pairs of (vertical-horizontal) segments to locate points in the square, where the sum of their lengths equals d (either d/2 + d/2 or d/4 + 3d/4). The student tends to rely on lower-level methods when faced with challenges in his/her activities, and the student reasoning process frequently jumps between levels 5 and 4. Therefore, his/her degree of acquisition of level 5 is considered *intermediate*.



It is going to be a square. It is clear that points A, B, C and D are equidistant from the centre. On the other hand, we can observe that AOD is an isosceles triangle (…) any point in its hypotenuse can be constructed as in the diagram with lengths b and h in the base and the height.

Since the triangle is isosceles, b+h=r (radius of the circumference in the Taxicab metric). See the right diagram (above). Then, the circumference is a square.

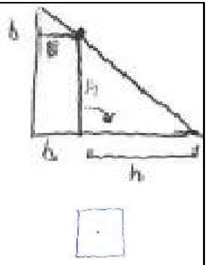
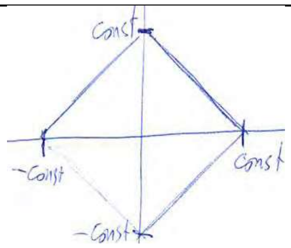


Figure 4: student #12 answer, level 5 – high acquisition

The student (see Figure 4) is trying to give a formal proof, as evidenced by the utilization of a chain of arguments grounded in mathematical principles. He/she can work with the Taxicab metric without recourse to the Euclidean metric. Notwithstanding, the conclusion reached incorporates an illustration of a square with horizontal and vertical sides, which suggests that the student incorrectly considers the possibility of rotating 45^0 a circumference when using the Taxicab metric. The student exhibits a consistent Level 5 reasoning capacity, albeit with some errors, thus warranting a classification of *high degree of acquisition*.



With this definition, the circumference will be given by $|x| + |y| = const$ or $|y| = -|y| = const$. Considering the 4 quadrants:

Q1: $y = -x + const$ Q2: $y = x + const$

Q3: $y = -x - const$ Q4: $y = x - const$

Thus, the circumference is a square-shaped rhombus with diagonals over the axis.

Figure 5: student #10 answer, level 5 – complete acquisition

The student's answer does not mention nor use any characteristic of the Euclidean circumference. Moreover, he/she understands and uses correctly the definition of the Taxicab metric and gets to express the distance in algebraic terms without mistakes. The student shows clear signs of a *complete degree of acquisition* of level 5, at least with respect to def2. It should be noted that, although the item focuses on the def2 indicator, the student's approach highlights other indicators, as he/she uses a definition of Taxicab metric equivalent to the given one for effectively addressing the task (def5).

Discussion

After analysing our data, we conclude that the item we designed is adequate for studying the acquisition of level 5, at least with respect to def2. The task was understood by all students, and most of them provided an actual answer. It does not require previous knowledge, then its answer is more related to the argumentation, which is a key aspect of the Van Hiele model. The answers indicate that level 5 features emerge when working with non-Euclidean metrics (Blair, 2004). It is noteworthy that the answers show different reasoning, which is desirable for assessing Van Hiele levels (Gutiérrez & Jaime, 1994). Specifically, the answer in Figure 3 (left) shows reasoning heavily influenced by knowledge of the Euclidean metric, whereas the answer in Figure 3 (right) is a mixture of reasoning that does not use the Taxicab metric during the whole process but recurs to the Euclidean metric at some points. Furthermore, some answers are constructed based solely on reasoning concerning the Taxicab metric. In addition, we observed different degrees of acquisition among the students.

Based on some of the answers, we believe that designing items that assess more than one indicator could be of great interest, as the task would be richer. However, such items would be more difficult to analyse. Thus, it is necessary to strike a balance between the different types of items. Finally, we could apply the questionnaire to Mathematics Undergraduates to obtain a larger number and a wider range of responses, as research-oriented graduates do not take this master's degree.

Further research is required to delve deeper into the distinctions between the different levels of acquisition, possibly by implementing the methodology proposed by Gutiérrez et al. (1991). In our work, we found differences between the answers related to argumentation, correctness, and completeness, which are key aspects of the Van Hiele model.

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